LAGRANGIAN EMBEDDINGS, MASLOV INDEXES AND INTEGER GRADED SYMPLECTIC FLOER COHOMOLOGY

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Dedicated to R. Fintushel on the occasion of his fifty-th birthday

Abstract. We define an integer graded symplectic Floer cohomology and a spectral sequence which are new invariants for monotone Lagrangian sub-manifolds and exact isotopies. Such an integer graded Floer cohomology is an integral lifting of the usual Floer-Oh cohomology with $Z_{\Sigma(L)}$ grading. As one of applications of the spectral sequence, we offer an affirmative answer to an Audin's question for oriented, embedded, monotone Lagrangian tori, i.e. $\Sigma(L) = 2$.

1. Introduction

One of important problems in symplectic topology is to understand when a manifold admits a Lagrangian embedding into C^n. M. Audin in [2] gave a series of obstructions to existence of such embeddings from purely topological point of view. In [2], she asked whether the following is true or not.

Audin's Question: Any embedded Lagrangian torus L in C^n has $\Sigma(L) = 2$, where $\Sigma(L)$ is the positively minimal Maslov number of L, see §2 for definition.

C. Viterbo in [34] derived a new obstruction to embedded Lagrangian tori, $2 \leq \Sigma(T^n) \leq n+1$. In particular, this answers the Audin's question for $n = 2$ since $\Sigma(L)$ is even for an oriented Lagrangian sub-manifold L. Viterbo used the symplectic action and Conley-Zehnder's finite dimensional reduction technique to relate the Maslov index with the Morse index in [33]. His method for T^n also works for compact manifold admitting a Riemannian metric with non-positive sectional curvature. In [26], [27], L. Polterovich obtained the same obstruction on the Maslov index $1 \leq \Sigma(L) \leq n+1$ for more general manifold L via Gromov's pseudo-holomorphic curves.

A new approach to the Lagrangian *rigidity* (relative to Lagrangian immersions, Gromov's h-principle, *flexibility* [3]) is given by Y. G. Oh in [22]. Using one of basic steps that Floer proved an Arnold conjecture for monotone symplectic manifolds [8], Oh has constructed a *local* symplectic Floer (co)homology such that there exists an approximation from local to global symplectic Floer (co)homology by adding the holomorphic disks or the quantum effect. This leads to a spectral sequence which converges to the usual symplectic Floer (co)homology. Thus the optimal rigidity for compact, embedded, monotone Lagrangian sub-manifolds is obtained in Theorem I [22], $1 \leq \Sigma(L) \leq$

n. There are examples in [27] which shows that the inequality $1 \leq \Sigma(L) \leq n$ can not be improved in general. Furthermore, Oh gives a positive answer to the Audin's question for monotone Lagrangian tori of dimension ≤ 24 (Theorem III [22]).

From understanding Oh's result [22] and [23], and Chekanov's result in [5], we define an integer graded symplectic Floer cohomology which can be thought as a combination of Oh's and Chekanov's approaches. In particular, we would like to replace the local symplectic Floer cohomology by a more global symplectic invariant, and to relate the restricted Floer (co)homology constructed in [5]. This leads us to construct an integer graded symplectic Floer cohomology in this paper which is *global* and integral lifting of the usual symplectic Floer cohomology, just like in the instanton theory developed by Fintushel and Stern [11]. For special Lagrangian sub-manifolds (representation varieties of handlebodies of an integral homology 3-sphere), Atiyah conjectured that the cohomologies from the instanton theory and the symplectic theory are same. The Atiyah's conjecture is proved in [18] by R. Lee and the author. This leads us to borrow some ideas from the instanton Floer theory. Our construction of the integer graded Floer cohomology is related to Chekanov's construction and Oh's construction via the Hofer's symplectic energy (see §5). This may give an interesting way to understand the Hofer's symplectic energy through the integer graded Floer cohomology.

Let (P, ω) be a monotone symplectic manifold and L is a monotone Lagrangian sub-manifold in (P, ω). Basically, we associate to L a discrete set $Ima_\phi(Z_\phi)$, and for each $r \in \mathbf{R} \setminus Ima_\phi(Z_\phi) = \mathbf{R}_{L,\phi}$ an Z_2-modules $I_*^{(r)}(L, \phi; P)$ with a natural integer grading. We also deal with more general case (L_1, L_2) in [19] to get an integer graded symplectic Floer homology for a pair of monotone Lagrangian sub-manifolds. These Z_2-modules will depend on r only through the interval in $\mathbf{R}_{L,\phi}$:

(i) $[r_0, r_1] \subset \mathbf{R}_{L,\phi}$, then $I_*^{(r_0)}(L, \phi; P) = I_*^{(r_1)}(L, \phi; P)$;

(ii) $I_{*+\Sigma(L)}^{(r)}(L, \phi; P) = I_*^{(r+\sigma(L))}(L, \phi; P)$, where $\sigma(L)(> 0)$ is the minimal number in $ImI_\omega|_{\pi_2(P,L)}$.

Theorem A. *For $\Sigma(L) \geq 3$, L is a monotone Lagrangian sub-manifold in (P, ω),*

(1) For any continuation $(J^\lambda, \phi^\lambda) \in \mathcal{P}_1$ which is regular at the ends, there exists an isomorphism

$$\phi_{02}^n : I_n^{(r)}(L, \phi^0; P, J^0) \to I_n^{(r)}(L, \phi^1; P, J^1),$$

for all $n \in \mathbf{Z}$.

(2) There is a spectral sequence $(E_{n,j}^k, d^k)$ with

$$E_{n,j}^1(L, \phi; P, J) \cong I_j^{(r)}(L, \phi; P, J), \quad n \equiv j \pmod{\Sigma(L)},$$

and

$$E_{n,j}^\infty(L, \phi; P, J) \cong F_n^{(r)} HF^j(L, \phi; P, J)/F_{n+\Sigma(L)}^{(r)} HF^j(L, \phi; P, J).$$

(3) The spectral sequence $(E_{n,j}^k, d^k)$ converges to the $Z_{\Sigma(L)}$ graded symplectic Floer cohomology $HF_*(L, \phi; P, J)$, where

$$d^k : E_{n,j}^k(L, \phi; P, J) \to E_{n+\Sigma(L)k+1,j+1}^k(L, \phi; P, J).$$

Theorem B. *For each $k \geq 1$, $E^k_{n,j}(L, \phi; P, J)$ are the symplectic invariant under continuous deformations of $(J^\lambda, \phi^\lambda)$ within in the set of continuations.*

All the $E^k_{n,j}(L, \phi; P, J) = E^k_{n,j}(L, \phi; P)$, $E^1_{n,j}(L, \phi; P) = I^{(r)}_n(L, \phi; P)$, for $k \geq 1, r \in \mathbf{R}_{L,\phi}$, are new symplectic invariants. They provided potentially interesting invariants for the symplectic topology of L. In particular the minimal k for which $E^k_{*,*} = E^\infty_{*,*}$ should be meaningful, denoted by $k(L)$. Using the integer graded Floer cohomology, the spectral sequence, and the new invariant $k(L)$, we give an affirmative answer to Audin's question for monotone Lagrangian embedding torus in C^m.

Theorem C. *For an embedded, oriented, monotone Lagrangian torus L in C^m, we have*

$$\Sigma(L) = 2.$$

The question remains open for *non-monotone* Lagrangian torus of dimension ≥ 3. We use the full information about the spectral sequence and some counting arguments to get the Theorem C unless $k(L) = \frac{m+1}{\Sigma(L)}$ (Theorem 5.11). Since the integer $\Sigma(L)$ is always even, so we first answer the Audin's question for all even dimensional monotone embedded torus. Hence the complete proof of Theorem C follows by studying for the odd dimensional case (Proposition 5.12). Note that Oh in [22] answered the Audin's question for monotone Lagrangian L with $m \leq 24$ (Theorem III in [22]). Our proof of Theorem C indicates the interaction between $k(L)$ and $\Sigma(L)$ for a monotone Lagrangian sub-manifold L. We also discuss the relation between the integer graded symplectic Floer cohomology and the Chekanov's construction [5]. Thus we obtain Chekanov's result by using the integer graded symplectic Floer cohomology. From these applications, the integer graded symplectic Floer cohomology plays a uniform role in the work of Oh [22] and the work of Chekanov [5]. This is why we use $\sigma(L)$ to restrict an energy band and $\Sigma(L) \geq 3$ to preserve the invariance from the definition of $I^{(r)}_*(L, \phi; P)$. Our study suggests a possible relation between the integer graded symplectic Floer cohomology and Hofer's symplectic energy for monotone Lagrangian sub-manifolds. In fact we conjecture that the Hofer's symplectic energy of a monotone Lagrangian sub-manifold L with $\Sigma(L) \geq 3$ is a positive multiple of $\sigma(L)$ (More precisely, $e_H(L) = k(L)\sigma(L)$), where $\sigma(L)$ is the minimal symplectic action on $\pi_2(P, L)$. We will discuss this problem elsewhere. It would be also interesting to link the integer graded symplectic Floer cohomology with the (modified) Floer cohomology with Novikov ring coefficients [15], [25].

The paper is organized as follows. In §2, we define the integer graded symplectic Floer cohomology for transversal Lagrangian intersections. Its invariant property under the symplectic continuations is given in §3. Theorem A (1) is proved in §3. Theorem A (2), (3) and Theorem B are proved in §4. In §5, we give some applications included Theorem C and relations among the Hofer's energy, the Chekanov's construction and the Oh's results.

2. Integer graded Floer cohomology for transversal Lagrangian intersections

Let (P, ω) be a compact (or tamed) symplectic manifold with a closed non-degenerate 2-form ω. The 2-form ω defines a second cohomology class $[\omega] \in H^2(P, \mathbf{R})$. By choosing an almost complex structure J on (P, ω) such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric, we have an integer valued second cohomology class $c_1(P) \in H^2(P, Z)$ the first Chern class. These two cohomology classes define two homomorphisms

$$I_\omega : \pi_2(P) \to \mathbf{R}; \qquad I_{c_1} : \pi_2(P) \to \mathbf{Z}.$$

If $u : (D^2, \partial D^2) \to (P, L)$ is a smooth map of pairs, there is a unique trivialization up to homotopy of the pull-back bundle $u^*TP \cong D^2 \times C^n$ as a symplectic vector bundle. This trivialization defines a map from $S^1 = \partial D^2$ to $\Lambda(C^n)$ the set of Lagrangians in C^n. Let $\mu \in H^1(\Lambda(C^n), \mathbf{Z})$ be the well-known Maslov class. Then we define a map

$$I_{\mu,L} : \pi_2(P, L) \to \mathbf{Z},$$

by $I_{\mu,L}(u) = \mu(\partial D^2)$, this Maslov index is invariant under any symplectic isotopy of P.

Definition 2.1. (i) (P, ω) is a monotone symplectic manifold if

$$I_\omega = \alpha I_{c_1}, \quad \text{for some } \alpha \geq 0$$

(ii) A Lagrangian sub-manifold L on P is monotone if

$$I_\omega = \lambda I_{\mu,L}, \quad \text{for some } \lambda \geq 0.$$

Remark: The monotonicity is preserved under the exact deformations of L. By the canonical homomorphism $f : \pi_2(P) \to \pi_2(P, L)$, one has

$$I_\omega(x) = I_\omega(f(x)), \qquad I_{\mu,L}(f(x)) = 2I_{c_1}(x),$$

where $x \neq 0 \in \pi_2(P)$. Therefore if L is a monotone Lagrangian sub-manifold, then P must be a monotone symplectic manifold and $2\lambda = \alpha$. In fact the constant λ does not depend on the Lagrangian L, but on (P, ω) if $I_\omega|_{\pi_2(P)} \neq 0$. For $\alpha = 0, \lambda = 0$ cases, they are monotone in the Floer's sense [7], [8]. The definition of monotone Lagrangian sub-manifolds is given by Oh in [21].

Definition 2.2. (i) Define $\sigma(L)$ be the positive minimal number in $ImI_\omega|_{\pi_2(P,L)} \subset \mathbf{R}$; Define $\Sigma(L)$ be the positive generator for the subgroup $[\mu|_{\pi_2(P,L)}] = ImI_{\mu,L}$ in $\mathbf{Z}$.

(ii) A Lagrangian sub-manifold L is called rational if $ImI_\omega|_{\pi_2(P,L)} = \sigma(L)\mathbf{Z}$ is a discrete subgroup of $\mathbf{R}$ and $\sigma(L) > 0$ [29]. For a monotone Lagrangian, we have $\sigma(L) = \lambda\Sigma(L)$ for some $\lambda > 0$.

Let $H : P \times \mathbf{R} \to \mathbf{R}$ be a smooth real valued function and let X_H be defined by $\omega(X_H, \cdot) = dH$. Then the ordinary differential equation

$$\frac{dx}{dt} = X_H(x(t)), \tag{2.1}$$

is called an Hamiltonian equation associated with the time-dependent Hamiltonian function H or the Hamiltonian vector field X_H. It defines a family of diffeomorphisms of P such that $x(t) = \phi_{H,t}(x)$ solves equation (2.1) for every x. A set $\mathcal{D}_\omega = \{\phi_{H,1} | H \in C^\infty(P \times \mathbf{R}, \mathbf{R})\}$ of all diffeomorphisms arising in this way is called the set of exact diffeomorphisms, an element in the set is called an exact isotopy.

Definition 2.3. For a given exact isotopy $\phi = \{\phi_t\}_{0 \le t \le 1}$ on (P, ω), we define

$$\Omega_\phi = \{z : I \to P \quad | \quad z(0) \in L, z(1) \in \phi_1(L), [\phi_t^{-1} z(t)] = 0 \in \pi_1(P, L)\}.$$

Proposition 2.4 (Oh [21])**.** *Let L be monotone and $\phi = \{\phi_t\}_{0 \le t \le 1}$ be an exact isotopy on (P, ω). Let u and v be two maps from $[0,1] \times [0,1]$ to Ω_ϕ such that*

$$u(\tau, 0), v(\tau, 0) \in L, \quad u(\tau, 1), v(\tau, 1) \in \phi(L)$$

$$u(0,t) = v(0,t) \equiv x, \quad u(1,t) = v(1,t) \equiv y, \quad x, y \in L \cap \phi(L).$$

Then we have

$$I_\omega(u) = I_\omega(v) \quad \textit{if and only if} \quad \mu_u(x,y) = \mu_v(x,y),$$

where μ_u is the Maslov-Viterbo index (see [9] *and* [33] *for definition).*

In particular, if u, v are J-holomorphic curve with respect to an almost complex structure J (may vary with time t) compatible with ω, then

$$\int \|\nabla u\|_J^2 = \int \|\nabla v\|_J^2 \quad \textit{if and only if} \quad \mu_u(x,y) = \mu_v(x,y).$$

This is Proposition 2.10 in [21]. Note that $\mu_u(x,y)$ is well-defined mod $\Sigma(L)$ (see Lemma 4.7 [21]).

The tangent space $T_z\Omega_\phi$ consists of vector fields ξ of P along z which are tangent to L at 0 and to $\phi(L)$ at 1. Then ω induces a "1-form" on Ω_ϕ.

$$Da(z)\xi = \int_{S^1} \omega(\frac{dz}{dt}, \xi(t))dt. \tag{2.2}$$

This form is closed in the sense that it can be integrated locally to a real function a on Ω_ϕ. It clearly vanishes for all ξ if and only if z is a constant loop, i.e. $z(0)$ is a fixed point of ϕ; its critical points Z_ϕ are the intersection points $L \cap \phi(L)$. A critical point is non-degenerate if and only if the corresponding intersection is transversal.

For a monotone Lagrangian sub-manifold L, an exact isotopy ϕ and $k > 2/p$, consider the space of L_k^p-paths

$$\mathcal{P}_{k,loc}^p(L, \phi; P) = \{u \in L_{k,loc}^p(\Theta, P) \ | \ u(\mathbf{R} \times \{0\}) \subset L, \ \ u(\mathbf{R} \times \{1\}) \subset \phi(L)\},$$

where $\Theta = \mathbf{R} \times [0,1] = \mathbf{R} \times iI \subset C$. Let S_ω be the bundle of all $J \in End(TP)$ whose fiber is given by

$$S_x = \{J \in End(T_xP) \ | \ J^2 = -Id \ \text{ and } \ \omega(\cdot, J\cdot) \ \text{ is a Riemannian metric}\},$$

and we denote the set of time-dependent almost complex structures by $\mathcal{J} = C^\infty([0,1] \times S_\omega)$. Define

$$\overline{\partial}_J u(\tau, t) = \frac{\partial u(\tau, t)}{\partial \tau} + J_t \frac{\partial u(\tau, t)}{\partial t}, \tag{2.3}$$

on $\mathcal{P}^p_{k,loc}$ and then the equation $\overline{\partial}_J u = 0$ is translationally invariant in the variable τ. Denote $\mathcal{M} = \mathcal{M}_J(L, \phi) = \{u \in \Omega_\phi | \int_{\mathbf{R}\times\mathbf{I}} |\frac{\partial u}{\partial \tau}|^2 < \infty\}$ and $\mathcal{M}(x,y) = \{u \in \mathcal{M}| \quad \lim_{\tau\to+\infty} u = x, \lim_{\tau\to-\infty} u = y; \quad x, y \in Z_\phi\}$.

$\mathcal{M} = \bigcup_{x,y\in L\cap\phi(L)} \mathcal{M}(x,y)$. If L intersects $\phi(L)$ transversely, then for each $x, y \in Z_\phi$, there exist smooth Banach manifold $\mathcal{P}(x,y) = \mathcal{P}^p_k(x,y) \subset \mathcal{P}^p_{k,loc}$ such that equation 2.2 defines a smooth section $\overline{\partial}_J$ of a smooth Banach space bundle $\mathcal{L}$ over $\mathcal{P}(x,y)$ with fibers $\mathcal{L}_u = L^p_{k-1}(u^*TP)$, and so that $\mathcal{M}(x,y)$ is the zero set of $\overline{\partial}_J$. The tangent space $T_u\mathcal{P}$ consists of all elements $\xi \in L^p_k(u^*(TP))$ so that $\xi(\tau, 0) \in TL, \xi(\tau, 1) \in T(\phi(L))$ for all $\tau \in \mathbf{R}$. The linearizations

$$E_u = D\overline{\partial}_J(u) : T_u\mathcal{P} \to \mathcal{L}_u \tag{2.4}$$

are Fredholm operators for $u \in \mathcal{M}(x,y)$. There is a dense set $\mathcal{J}_{reg}(L, \phi(L)) \subset \mathcal{J}$ so that if $J \in \mathcal{J}_{reg}(L, \phi(L))$, then E_u is surjective for all $u \in \mathcal{M}(x,y)$. Moreover the Fredholm index of the linearization $Ind(E_u)$ is the same as the Maslov-Viterbo index $\mu_u(x,y)$. In particular $\mathcal{M}_J(x,y)$ becomes a smooth manifold with dimension equal to $\mu_u(x,y)$ for $J \in \mathcal{J}_{reg}(L, \phi(L))$. See Proposition 2.1 [7] and Theorem 1 [9].

Theorem 2.5 (Floer [7] and Oh [21] Theorem 4.6)**.** *Let L be a monotone Lagrangian sub-manifold in P and $\phi = \{\phi_t\}_{0\le t\le 1}$ be an exact isotopy such that L is transversally intersects $\phi_1(L)$. Let $C_*(L, \phi; P, J)$ be the free generated Z_2-modules from Z_ϕ. Suppose $\Sigma(L) \geq 3$. Then there exists a homomorphism*

$$\delta : C_*(L, \phi; P, J) \to C_*(L, \phi; P, J) \tag{2.5}$$

with $\delta \circ \delta = 0$ for $J \in \mathcal{J}_{reg}(L, \phi(L))$. Define the Floer cohomology $HF^*_J(L, \phi; P, J)$ as the cohomology of $H^*(C_*(L, \phi; P, J), \delta)$, a $\mathbf{Z}/\Sigma(L)$-graded Z_2-modules. $HF^*_J(L, \phi; P, J)$ is invariant under the continuation of (J, ϕ), denoted by $HF^*(L, \phi; P)$ with $* \in Z_{\Sigma(L)}$.

In order to extend this Floer-Oh theory to one with an integer grading we make use of an infinite cyclic cover $\tilde{\Omega}_\phi$ of Ω_ϕ so that the symplectic action on $\tilde{\Omega}_\phi$ and the Maslov index function on $\tilde{Z}_\phi$ define as $a : \tilde{\Omega}_\phi \to \mathbf{R}$ and $\mu : \tilde{Z}_\phi \to \mathbf{Z}$. From (2.2), a functional a on Ω_ϕ is only defined up to $\sigma(L)Z$, i.e. $a : \Omega_\phi \to \mathbf{R}/\sigma(L)Z$ from the different topology classes in $\pi_2(P, L)$. The functional a on Ω_ϕ and its lift on $\tilde{\Omega}_\phi$ are clearly distinguished from its context.

Lemma 2.6. *There exists a universal covering space $\tilde{\Omega}_\phi$ of Ω_ϕ with transformation group $\pi_2(P, L)$.*

Proof: The function space Ω_ϕ has the homotopy type of a CW complex and so an associated universal covering space. By Milnor's theorem 3.1 in [20], there is a universal covering space $\tilde{\Omega}_\phi$ of Ω_ϕ. By definition of Ω_ϕ, there is a homotopy $F(\tau, t)$ of $\phi_t^{-1}u(\tau, t)$ to a constant path in (P, L) for any $u : I \to \Omega_\phi$. Thus we can redefine the map u to yield a map $\overline{u}(\tau, t) = u(2\tau, t)$ for $0 \le \tau \le 1/2$; and $\overline{u}(\tau, t) = F(2\tau - 1, t)$ for $1/2 \le \tau \le 1$. Such a map $\overline{u} : (D^2, \partial D^2) \to (P, L)$ defines an element in

$\pi_2(P, L)$. It is easy to check that $u \to \overline{u}$ is a bijective homomorphism between $\pi_1(\Omega_\phi)$ and $\pi_2(P, L)$ (see also Proposition 2.3 in [7]). $\square$

Now the closed 1-form $Da(z)$ has a function up to a constant such that $a : \tilde{\Omega}_\phi \to \mathbf{R}$ is well-defined. Pick a point $z_0 \in L \cap \phi(L)$ such that $a(z_0) = 0$ by adding a constant. For $g \in \pi_1(\Omega_\phi) = \pi_2(P, L)$ we have

$$a(g(x)) = a(x) + \deg(g)\sigma(L), \tag{2.6}$$

where $\deg(g)$ is defined as $I_\omega(g) = \deg(g)\sigma(L)$. Let $Z_\phi = \{x \in L \cap \phi(L) \ \mid \ [\phi_t(x)] = 0 \in \pi_1(P, L)\}$. Let $Ima(Z_\phi)$ be the image of a of Z_ϕ; modulo $\sigma(L)\mathbf{Z}$, $Ima(Z_\phi)$ is a finite set. Thus a set $\mathbf{R}_{L,\phi} = \mathbf{R} \setminus Ima(Z_\phi)$ consists of the regular values of the symplectic action a on $\tilde{\Omega}_\phi$. In this section, we are going to construct an integer graded symplectic Floer cohomology for every $r \in \mathbf{R}_{L,\phi}$. Given $x \in Z_\phi \subset \Omega_\phi$, let $x^{(r)} \in \tilde{Z}_\phi \subset \tilde{\Omega}_\phi$ be the unique lift of x such that $a(x^{(r)}) \in (r, r + \sigma(L))$. Let $\mu^{(r)}(x) = \mu(x^{(r)}, z_0) \in \mathbf{Z}$ and define the integral symplectic Floer cochain group

$$C_n^{(r)}(L, \phi; P, J) = \mathbf{Z}_2\{x \in Z_\phi \quad | \quad \mu^{(r)}(x) = n \in \mathbf{Z}\}, \tag{2.7}$$

as the free Z_2 module generated by $x \in Z_\phi$ with the lift $x^{(r)}$ and $\mu(x^{(r)}, z_0) = n$. If $\overline{z}_0$ is another choice of a based point and $g(z_0) = \overline{z}_0$ for some covering transformation g, then the corresponding choice of lift $\overline{x}^{(r)}$ of x is just $g(x^{(r)})$. Note that the integral Maslov index $\mu_u^{(r)}(x)$ is independent of the choice of the based point z_0 used in the definition of a by (2.6). The following lemma shows that the lift of the functional a is compatible with a universal lift of $\mathbf{R}/\sigma(L)Z$.

Lemma 2.7. *The lift of the symplectic action over $\tilde{\Omega}_\phi$ is compatible with the one of the Maslov index, i.e. for $g \in \pi_2(P, L)$ with $deg(g) = n$,*

$$a(g(z_0)) = n\sigma(L) \quad \textit{if and only if} \quad \mu^{(r)}(g(z_0), z_0) = n\Sigma(L).$$

Proof: Let J be a compatible almost complex structure and $\omega(\cdot, J\cdot)$ be the corresponding Riemannian metric on P. Denote ∇ be the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$. Then T_xL is an orthogonal complement of JT_xL. One can represent J_x to be a standard J for suitable orthonormal basis in T_xL. Let h be a parallel transport along the path $u(\tau)$ ($u(\tau, t)$ for each fixed $t \in I$) in Ω_ϕ. Then we obtain an isometry

$$h_{\tau,t} : T_xP \to T_{u(\tau,t)}P.$$

Define $J_{\tau,t} = h_{\tau,t}^{-1} \circ J_{u(\tau,t)} \circ h_{\tau,t}$. Then we have a smooth map $f : I \times I \to SO(2m)$ such that $f_{\tau,t}^{-1} \circ J_{\tau,t} \circ f_{\tau,t} = J_x$. Set

$$\tilde{L}(\tau) = h_{\tau,0}^{-1}(T_{u(\tau,0)}L), \quad \tilde{\phi(L)}(\tau) = h_{\tau,1}^{-1}(T_{u(\tau,1)}\phi(L)).$$

Thus $f_{\tau,0}(\tilde{L}(\tau)) = L(\tau)$, $f_{\tau,1}(\tilde{\phi(L)}(\tau)) = \phi(L)(\tau)$. The trivialization of u^*TP by using the parallel transportation $\{h_{\tau,t}\}$ is given by

$$u^*(T_xP) = I \times I \times T_xP = I \times I \times C^m.$$

Then there are two paths of Lagrangian subspaces $L(\tau)$ and $\phi(L)(\tau)$ in $T_xP = C^m$. Note that these two Lagrangian paths intersects transversally at end points $\tau = 0, 1$. There is a map f_ϕ from Ω_ϕ to

the space of pairs of Lagrangian subspaces $\Lambda(m)$ which has a universal covering $\tilde{\Lambda}(m)$ (see [1], [4] §1), where $f_\phi(\{u(\tau)\}) = \{L(\tau), \phi(L)(\tau)\}, 0 \leq \tau \leq 1$. For the map f_ϕ from Ω_ϕ to $\Lambda(m)$, there is a map from the CW complex Ω_ϕ to $\Lambda(m)$ from the obstruction theory. Hence there exists a corresponding map F between the universal covering space $\tilde{\Omega}_\phi$ and the universal covering space $\tilde{\Lambda}(m)$. From the choice of z_0, we have $a(g(z_0)) = n\sigma(L)$. Here we have $u(0) = z_0, u(1) = g(z_0)$ and $\{u(\tau)\}_{0\leq\tau\leq 1}$ corresponding to $g \in \pi_2(P, L)$.

Note that we have deg : $\pi_1(\Omega_\phi) \to \sigma(L)\mathbf{Z}$ and $Mas : \pi_1(\Lambda(m)) \cong \Sigma(L)\mathbf{Z}$ for our case. Hence there is an induced $g_1 \in \pi_1(\Lambda(m))$ such that $g_1 F = f_\phi g$. $\mu^{(r)}(z_0, z_0) = 0$. We have the following commutative diagram

$$\begin{array}{ccc} \pi_1(\Omega_\phi) & \xrightarrow{\pi_1(f_\phi)} & \pi_1(\Lambda(m)) \\ \downarrow deg(g) & & \downarrow deg(g_1) \\ \sigma(L)\mathbf{Z} & \xrightarrow{F_*} & \Sigma(L)\mathbf{Z}. \end{array}$$

So $I_\omega(u_g) = n\sigma(L), I_{\mu,L}(u_g) = deg(g_1)\Sigma(L)$ by the definitions of $\Sigma(L)$ and the Maslov index (see [1]). Thus the result follows from the monotonicity and $\sigma(L) = \lambda\Sigma(L)$. □

Remark: The index $\mu_u(x)$ depends on the trivialization over $I \times I$, only the relative index does not depend on the trivialization. So the choice of a single z_0 fixes the shifting in the integer graded Floer cochain complex.

Definition 2.8. The integral Floer coboundary map $\partial^{(r)} : C^{(r)}_{n-1}(L, \phi; P, J) \to C^{(r)}_n(L, \phi; P, J)$ is defined by

$$\partial^{(r)} x = \sum_{y \in C^{(r)}_n(L,\phi;P,J)} \#\hat{\mathcal{M}}(x, y) \cdot y,$$

where $\mathcal{M}(x, y)$ denote the union of the components of 1-dimensional moduli space of J-holomorphic curves and $\hat{\mathcal{M}}(x, y) = \mathcal{M}(x, y)/\mathbf{R}$ is a zero dimensional moduli space modulo τ-translationally invariant. $\#\hat{\mathcal{M}}(x, y)$ counts the points modulo 2.

Note that the coboundary map $\partial^{(r)}$ only counts part of the Floer-Oh's coboundary map in [21]. We are going to show that $\partial^{(r)} \circ \partial^{(r)} = 0$. The corresponding cohomology groups are the integer graded symplectic Floer cohomology $I^{(r)}_*(L, \phi; P, J), * \in \mathbf{Z}$.

Proposition 2.9. *If $u \in \mathcal{P}(x, y)$ for $x, y \in Z_\phi$ and $\tilde{u}$ is any lift of u, then $\mu_{\tilde{u}} = \mu^{(r)}(y) - \mu^{(r)}(x)$.*

This follows from Lemma 2.7 or Proposition 2.4 [7] and Proposition 2.10 [21].

Proposition 2.10 (Oh , Proposition 4.1 and 4.3 [21])**.** *Suppose that two Lagrangian sub-manifolds L and $\phi(L)$ intersect transversally and $\Sigma(L) \geq 3$. Then there is a dense subset $\mathcal{J}_*(L, \phi) \subset \mathcal{J}_{reg}(L, \phi)$ of $\mathcal{J}$ such that (1) the zero dimensional component of $\hat{\mathcal{M}}(x, y)$ is compact and (2) the one dimensional component of $\hat{\mathcal{M}}(x', y')$ is compact up to the splitting of two isolated trajectories for $J \in \mathcal{J}_*(L, \phi)$.*

Proposition 2.10 plays the key role in showing that $\delta \circ \delta = 0$ in Theorem 2.5. We follow Floer-Oh's argument to show that $\partial^{(r)} \circ \partial^{(r)} = 0$. The condition $\Sigma(L) \geq 3$, rather than $\Sigma(L) \geq 2$,

enters only in proving that $< \delta \circ \delta x, x >= 0$. For $\Sigma(L) = 2$, Oh evaluates a number $\pmod 2$ of J-holomorphic discs with Maslov index 2 that pass through $x \in L \subset P$ in [24]. Oh in [24] verified that the number is always even, hence $< \delta \circ \delta x, x >= 0 \pmod 2$. In our case, this reflects to understand the two lifts $x^{(r)}, g(x^{(r)})$ of x with $deg(g) = \pm 1$. Note that $x^{(r)} \in (r, r + \sigma(L))$, $g(x^{(r)}) \in (r + deg(g)\sigma(L), r + (deg(g) + 1)\sigma(L))$. The integer graded symplectic coboundary is not well-defined in this situation. We leave it to a future study.

Lemma 2.11. *Under the same hypothesis in Proposition 2.10,* $\partial^{(r)} \circ \partial^{(r)} = 0$.

Proof: If $x \in C^{(r)}_{n-1}(L, \phi; P, J)$, then by definition the coefficient of $z \in C^{(r)}_{n+1}(L, \phi; P, J)$ in $\partial^{(r)} \circ \partial^{(r)}(x)$ is

$$\sum_{y \in C^{(r)}_n(L,\phi;P,J)} \#\hat{\mathcal{M}}(x, y) \cdot \#\hat{\mathcal{M}}(y, z). \tag{2.8}$$

By Proposition 2.10, the boundary of the 1-dimensional manifold $\hat{\mathcal{M}}(x, z) = \mathcal{M}(x, z)/\mathbf{R}$ corresponds to two isolated trajectories $\mathcal{M}(x, y) \times \mathcal{M}(y, z)$. Each term $\#\hat{\mathcal{M}}(x, y) \cdot \#\hat{\mathcal{M}}(y, z)$ is the number of the 2-cusp trajectory of $\hat{\mathcal{M}}(x, z)$ with $y \in C^{(r)}_n(L, \phi; P, J)$. For any such y there are J-holomorphic curves $u \in \mathcal{M}(x, y)$ and $v \in \mathcal{M}(y, z)$. The other end of the corresponding component of the 1-manifold $\hat{\mathcal{M}}(x, z)$ corresponding to the splitting $\mathcal{M}(x, y^{'}) \times \mathcal{M}(y', z)$ with $u^{'} \in \mathcal{M}(x, y^{'})$ and $v^{'} \in \mathcal{M}(y', z)$. Then $\hat{\mathcal{M}}(x, z)$ has an one parameter family of paths from x to z with ends $u\#v$ and $u^{'}\#v^{'}$ for appropriate grafting [7]. If we lift u to $\tilde{u} \in \tilde{\mathcal{M}}(x^{(r)}, \tilde{y})$, then

$$1 = \mu_{\tilde{u}} = \mu^{(r)}(\tilde{y}) - \mu^{(r)}(x) = \mu^{(r)}(\tilde{y}) - (n - 1). \tag{2.9}$$

So $\mu^{(r)}(\tilde{y}) = n$; and $\tilde{y} = y^{(r)}$ is the preferred lift, thus we have $\tilde{u} \in \tilde{\mathcal{M}}(x^{(r)}, y^{(r)})$. Similarly $\tilde{v} \in \tilde{\mathcal{M}}(y^{(r)}, z^{(r)})$. Since $u^{'}\#v^{'}$ is homotopic to $u\#v$ rel $(x^{(r)}, z^{(r)})$, the lift $\tilde{u}^{'}\#\tilde{v}^{'}$ is also a path with ends $(x^{(r)}, z^{(r)})$. Now using the symplectic action a is non-increasing along the gradient trajectory $\tilde{u}^{'}$, we have

$$r < a(z^{(r)}) \leq a(\tilde{y}^{'}) \leq a(x^{(r)}) < r + \sigma(L). \tag{2.10}$$

By uniqueness, $\tilde{y}^{'} = (y^{'})^{(r)}$ and using (2.9) for $u^{'}$, we have $\mu^{(r)}((y^{'})^{(r)}) = \mu^{(r)}(x^{(r)}) + 1 = n$; so $(y^{'})^{(r)} \in C^{(r)}_n(L, \phi; P, J)$. Thus the number of two-trajectories connecting $x^{(r)}$ and $z^{(r)}$ with index 2 is even. □

Now $(C^{(r)}_n(L, \phi; P, J), \partial^{(r)}_n)_{n \in \mathbf{Z}}$ is indeed an integer graded Floer cochain complex. We call its cohomology to be an integer graded symplectic Floer cohomology, denote by

$$I^{(r)}_*(L, \phi; P, J) = H^*(C^{(r)}_*(L, \phi; P, J), \partial^{(r)}), \quad * \in \mathbf{Z}.$$

From the construction we see that if $[r, s] \subset \mathbf{R}_{L,\phi}$, then $I^{(r)}_*(L, \phi; P, J) = I^{(s)}_*(L, \phi; P, J)$. The relation between $I^{(r)}_*(L, \phi; P, J)$ and $HF^*(L, \phi; P)$ will be discussed in §4.

3. Invariance property of the integral symplectic Floer cohomology

In this section we are going to show that the integer graded symplectic Floer cohomology defined in the previous section is invariant under the change of J and under the exact deformations of Lagrangian sub-manifolds.

We consider an one parameter family $\{(J^\lambda, \phi^\lambda\}_{\lambda \in \mathbf{R}}$ that interpolates from (J^0, ϕ^0) to (J^1, ϕ^1) and is constant in λ outside $[0,1]$. Let $\mathcal{P}_{1,\varepsilon/2}$ be the set of all one parameter families where $J^0, J^1 \in \mathcal{J}_*(L, \phi) \subset \mathcal{J}_{reg}(L, \phi)$ such that

$$\int_{\Theta \times [0,1]} |(J^\lambda_\tau - J^i)\frac{\partial u_\lambda}{\partial t}|^2 dt d\tau < \frac{\varepsilon}{2}, \quad |\phi^\lambda - id|_{C^1(\Theta \times [0,1])} < \frac{\varepsilon}{2}, \tag{3.1}$$

for all $u_\lambda \in \mathcal{P}(L, \phi_\lambda; P)$, $i = 0, 1$. Here we also assume that ϕ^λ is exact under the change of λ, so $J^\lambda_t = J(\lambda, t), \phi^\lambda_t = \phi(\lambda, t), \phi(\lambda, 0) = Id$ are two parameter families of almost complex structures compatible to ω and exact isotopes contractible to the identity. Such ϕ^λ_t connecting ϕ^0, ϕ^1 does exist. Floer discussed the invariance of the Floer cohomology under the change of (J, ϕ) for (J^0, ϕ^0) C^∞-close to (J^1, ϕ^1) in [8]. For (J, ϕ), define

$$C^{(r)}_{J,L} = \min\{a(x^{(r)}) - r, \sigma(L) + r - a(x^{(r)}) | x \in Z_\phi\}.$$

Since $r \in \mathbf{R}_{L,\phi}$ is an regular value of the symplectic action and modulo $\sigma(L)\mathbf{Z}$ the image $Ima(Z_\phi)$ is a finite set, so we have $C^{(r)}_{J,L}$ is a positive number in $(0, \sigma(L))$. For instance, we may choose $\varepsilon = C^{(r)}_{J,L}/16$.

Define the perturbed J-holomorphic curve equations

$$\overline{\partial}_{J^\lambda} u_\lambda(\tau, t) = \frac{\partial u_\lambda}{\partial \tau} + J^\lambda_t \frac{\partial u_\lambda}{\partial t} = 0, \tag{3.2}$$

with the moving Lagrangian coboundary conditions

$$u_\lambda(\tau, 0) \in L, u_\lambda(\tau, 1) \in \phi^\lambda_1(L). \tag{3.3}$$

This directly generalizes the J-holomorphic curve equation in the case of (J^0, ϕ^0) and (J^1, ϕ^1). The moduli space $\mathcal{M}_\lambda(x, y)$ of (3.2) and (3.3) has the same analytic properties as the moduli space $\mathcal{M}(x, y)$ except for the translational invariance (see Proposition 3.2 in [7]). Hofer in [13] anaylized the compactness property for a similar moving Lagrangian coboundary condition, Oh in [21] studied that the bubbling-off a sphere or disk can not occur in the components of $\mathcal{M}_\lambda(x, y)$ for the monotone Lagrangian sub-manifold L and $\Sigma(L) \geq 3$. The index of u_λ can be proved to be the same as a topological index for the moduli space of perturbed J-holomorphic curves. The arguments of the proof of invariance under the change of (J, ϕ) are the same as in [7], [8] and [21]. But we need to ensure that the cochain map is well-defined for the integer graded cochain complexes.

Lemma 3.1. *If $u_\lambda \in \mathcal{M}_\lambda(x_0, x_1)$, $(J^\lambda_t, \phi^\lambda_t) \in \mathcal{P}_{1,\varepsilon/2}$ and $\tilde{u}_\lambda \in \mathcal{P}(\tilde{x}_0, \tilde{x}_1)$ is any lift of u_λ, then*

$$a_{(J^1,\phi^1)}(\tilde{x}_1) < a_{(J^0,\phi^0)}(\tilde{x}_0) + \varepsilon.$$

Proof: Note that the path $\{u_\lambda(\tau)|\tau \in (-\infty,0)\}$ is a gradient trajectory for (J^0,ϕ^0) and $\{u_\lambda(\tau)|\tau \in (1,\infty)\}$ is a gradient trajectory for (J^1,ϕ^1). So we have

$$a_{(J^0,\phi^0)}(\tilde{u}(0)) \leq a_{(J^0,\phi^0)}(\tilde{x}_0), \quad a_{(J^1,\phi^1)}(\tilde{x}_1) \leq a_{(J^1,\phi^1)}(\tilde{u}(1)). \tag{3.4}$$

Since $u_\lambda \in \mathcal{M}_\lambda(x_0,x_1)$, by the property of $\mathcal{P}_{1,\varepsilon/2}$, we have

$$\begin{aligned} I_\omega(u_\lambda)|_{\Theta\times[0,1]} &= \|\partial_{J^\lambda} u_\lambda\|^2_{L^2(\Theta\times[0,1])} - \|\overline{\partial}_{J^\lambda} u_\lambda\|^2_{L^2(\Theta\times[0,1])} \\ &\geq \|\partial_{J^0} u_\lambda\|^2_{L^2(\Theta\times[0,1])} - \|(\partial_{J^\lambda}-\partial_{J^0})u_\lambda\|^2_{L^2(\Theta\times[0,1])} \\ &\geq -\frac{1}{2}\|(J^\lambda_t - J^0)\frac{\partial u_\lambda}{\partial t}\|^2_{L^2(\Theta\times[0,1])} \\ &> -\frac{\varepsilon}{2} \end{aligned}$$

Note that $\partial_J u = \frac{1}{2}(\frac{\partial u}{\partial \tau} - J\frac{\partial u}{\partial t})$. So the symplectic actions for $\tilde{u}(0)$ and $\tilde{x}_1$ are related by the following.

$$a(\tilde{u}(0)) - a(\tilde{x}_1) = I_\omega(u_\lambda)|_{\Theta\times[0,1]} > -\frac{\varepsilon}{2}. \tag{3.5}$$

Thus from (3.4) and (3.5) we get

$$a_{(J^0,\phi^0)}(\tilde{x}_0) \geq a_{(J^0,\phi^0)}(\tilde{u}(0)) > a(\tilde{x}_1) - \frac{\varepsilon}{2}. \tag{3.6}$$

Then the result follows. □

Definition 3.2. For each n, define a homomorphism $\phi^n_{01} : C^{(r)}_n(L,\phi^0;P,J^0) \to C^{(r)}_n(L,\phi^1;P,J^1)$ by

$$\phi^n_{01}(x_0) = \sum_{x_1\in C^{(r)}_n(L,\phi^1;P,J^1)} \#\mathcal{M}^0_\lambda(x_0,x_1)\cdot x_1.$$

We show that the homomorphism ϕ^n_{01} is indeed a cochain map with respect to the integral lift r.

Lemma 3.3. *$\{\phi^*_{01}\}_{*\in Z}$ is a cochain map, i.e.*

$$\partial^{(r)}_{n,1}\circ\phi^n_{01} = \phi^n_{01}\circ\partial^{(r)}_{n,0},$$

for all $n \in Z$.

Proof: For $x_0 \in C^{(r)}_n(L,\phi^0;P,J^0)$ and $y_1 \in C^{(r)}_{n+1}(L,\phi^1;P,J^1)$,the coefficient of y_1 in $(\partial^{(r)}_{n,1}\circ\phi^n_{01} - \phi^n_{01}\circ\partial^{(r)}_{n,0})(x_0)$ is the number of the set modulo 2,

$$\cup_{y_0\in C^{(r)}_{n+1}(L,\phi^0;P,J^0)} -\hat{\mathcal{M}}_{J^0,\phi^0}(x_0,y_0)\times\mathcal{M}^0_\lambda(y_0,y_1) \cup_{x_1\in C^{(r)}_n(L,\phi^1;P,J^1)} \mathcal{M}^0_\lambda(x_0,x_1)\times\hat{\mathcal{M}}_{J^1,\phi^1}(x_1,y_1). \tag{3.7}$$

The ends of the 1-dimensional manifold $\mathcal{M}^1_\lambda(x_0,y_1)$ are in one-to-one correspondence with the set

$$\cup_{y\in Z_{\phi_0}} -\hat{\mathcal{M}}_{J^0,\phi^0}(x_0,y)\times\mathcal{M}^0_\lambda(y,y_1) \cup_{x\in Z_{\phi_1}} \mathcal{M}^0_\lambda(x_0,x)\times\hat{\mathcal{M}}_{J^1,\phi^1}(x,y_1). \tag{3.8}$$

For an end $u\#v$ of $\mathcal{M}^1_\lambda(x_0,y_1)$ corresponding to an element in (3.7), the other end of the component $u^{'}\#v^{'}$ corresponds to an element in (3.8). For $u \in \mathcal{M}^0_\lambda(x_0,y)$ and $v \in \hat{\mathcal{M}}^1_{J^1,\phi^1}(y,y_1)$, $\mathcal{M}^1_\lambda(x_0,y_1)$ gives a 1-parameter family of paths in $\mathcal{P}(L,\phi_\lambda;P)$ with fixed end points x_0,y_1. Such a path gives a homotopy of paths from $u\#_\rho v$ to $u^{'}\#_\rho v^{'}$ rel end points. The lift of $u\#_\rho v$ starting $x^{(r)}_0$ ending at

$y_1^{(r)}$, so the same is true for the lift of $u^{'}\#_\rho v^{'}$ starting at $x_0^{(r)}$. Suppose $u^{'}$ lifts to an element in $\mathcal{M}^0_\lambda(x_0^{(r)}, \tilde{y})$. By Lemma 3.1, we have

$$a_{(J^1,\phi^1)}(\tilde{y}) < a_{(J^0,\phi^0)}(x_0^{(r)}) + \varepsilon < r + \sigma(L).$$

Furthermore, using the trajectory decreasing the symplectic action, we also have

$$a_{(J^1,\phi^1)}(\tilde{y}) > a_{(J^1,\phi^1)}(y_1^{(r)}) > r.$$

So this shows that $a_{(J^1,\phi^1)}(\tilde{y}) \in (r, r+\sigma(L))$ which gives the preferred lift $\tilde{y} = y^{(r)}$. Now using Proposition 2.9,

$$1 = \mu^{(r)}(y_1^{(r)}) - \mu^{(r)}(y^{(r)}) = (n+1) - \mu^{(r)}(y^{(r)}),$$

so we get $\mu^{(r)}(y^{(r)}) = n$, $y \in C_n^{(r)}(L, \phi^1; P, J^1)$. This shows that $u^{'}\#_\rho v^{'}$ in (3.8) actually corresponds to an element in (3.7). So the cardinality is always even. $\square$

For $(J^i, \phi^i) \in \mathcal{P}_1$, $i = 0, 1, 2$, and define new classes of perturbations $\mathcal{P}_{2,\varepsilon}$ to consists of

$$(J^\lambda, \phi^\lambda) = \begin{cases} (J^0, \phi^0) & \lambda \leq -T \\ (J^1, \phi^1) & -T+1 \leq \lambda \leq T-1 \\ (J^2, \phi^2) & \lambda \geq T \end{cases}$$

for some fixed positive number $T(>2)$, such that

$$\int_{\Theta \times ([-T,-T+1] \cup [T-1,T])} |(J^\lambda - J^i)\frac{\partial u_\lambda}{\partial t}|^2 dt d\tau < \varepsilon. \tag{3.9}$$

Then if $(J_1^\lambda, \phi_1^\lambda) \in \mathcal{P}_{1,\varepsilon/2}((J^0, \phi^0), (J^1, \phi^1))$ and $(J_2^\lambda, \phi_2^\lambda) \in \mathcal{P}_{1,\varepsilon/2}((J^1, \phi^1), (J^2, \phi^2))$, then we can compose $(J_1^\lambda, \phi_1^\lambda)$ with $(J_2^\lambda, \phi_2^\lambda)$ to get $(J^\lambda, \phi^\lambda) \in \mathcal{P}_{2,\varepsilon}((J^0, \phi^0), (J^2, \phi^2))$. We denote such a composition as $(J^\lambda, \phi^\lambda) = (J_1^\lambda, \phi_1^\lambda)\#_T(J_2^\lambda, \phi_2^\lambda)$. Then for a large fixed T and each compact set K in $\mathcal{M}_{(J_1^\lambda, \phi_1^\lambda)}(x, y) \times \mathcal{M}_{(J_2^\lambda, \phi_2^\lambda)}(y, z)$, there is a $\rho_T > 0$ and for all $\rho > \rho_T$ a local diffeomorphism

$$\# : \mathcal{M}_{(J_1^\lambda, \phi_1^\lambda)}(x, y) \times \mathcal{M}_{(J_2^\lambda, \phi_2^\lambda)}(y, z) \supset K \to \mathcal{M}_{(J_1^\lambda, \phi_1^\lambda)\#_T(J_2^\lambda, \phi_2^\lambda)}(x, z).$$

See Proposition 2d.1 in [8].

Lemma 3.4. *For the above* $(J^\lambda, \phi^\lambda)$ *and* $\rho > \rho_T$*,*

$$\phi_{02}^n = \phi_{12}^n \circ \phi_{01}^n,$$

for all $n \in \mathbf{Z}$.

Proof: For $x_0 \in C_n^{(r)}(L, \phi^0; P, J^0)$,

$$\phi_{02}^n(x_0) = \sum_{y_0 \in C_n^{(r)}(L, \phi^2; P, \mathcal{J}^\in)} \#\mathcal{M}^0_\lambda(x_0, y_0) \cdot y_0,$$

$$\phi_{12}^n \circ \phi_{01}^n(x_0) = \sum \#(\mathcal{M}^0_{(J_1^\lambda, \phi_1^\lambda)}(x_0, y) \times \mathcal{M}_{(J_2^\lambda, \phi_2^\lambda)}(y, y_0)) \cdot y_0,$$

where the summation $\sum$ runs over $y \in C_n^{(r)}(L, \phi^1; P, J^1)$ and $y_0 \in C_n^{(r)}(L, \phi^2; P, J^2)$. The local diffeomorphism $\#$ implies that

$$\#\mathcal{M}^0_\lambda(x_0, y_0) = \#(\mathcal{M}^0_{(J_1^\lambda, \phi_1^\lambda)}(x_0, y) \times \mathcal{M}^0_{(J_2^\lambda, \phi_2^\lambda)}(y, y_0)).$$

All we need to check is that $y \in C_n^{(r)}(L, \phi^1; P, J^1)$, this can be checked by the same argument in Lemma 3.3. $\square$

For two classes $(J^\lambda, \phi^\lambda)$, $(\overline{J}^\lambda, \overline{\phi}^\lambda)$ in $\mathcal{P}_{2,\varepsilon}((J^0, \phi^0), (J^2, \phi^2))$, the following lemma shows that the induced cochain maps ϕ_{02}^n, $\overline{\phi}_{02}^n$ are cochain homotopic to each other.

Lemma 3.5. *If $(J^\lambda, \phi^\lambda)$, $(\overline{J}^\lambda, \overline{\phi}^\lambda)$ in $\mathcal{P}_{2,\varepsilon}((J^0, \phi^0), (J^2, \phi^2))$ can be smoothly deformed from one to another by a 1-parameter family $(J_s^\lambda, \phi_s^\lambda), s \in [0,1]$, i.e. $(J_s^\lambda, \phi_s^\lambda) = (J^\lambda, \phi^\lambda)$ for $s \leq 0$ and $(J_s^\lambda, \phi_s^\lambda) = (\overline{J}^\lambda, \overline{\phi}^\lambda)$ for $s \geq 1$. Then ϕ_{02}^*, $\overline{\phi}_{02}^*$ are cochain homotopic to each other.*

Proof: we need to construct a homomorphism

$$H : C_*^{(r)}(L, \phi^0; P, J^0) \to C_*^{(r)}(L, \phi^2; P, J^2),$$

of degree -1 with the property

$$\phi_{02}^n - \overline{\phi}_{02}^n = H\partial_{n,0}^{(r)} + \partial_{n,2}^{(r)} H, \tag{3.10}$$

for all $n \in \mathbf{Z}$. Associated to $(J_s^\lambda, \phi_s^\lambda)$, there is a moduli space $H\mathcal{M}(x_0, y_0) = \cup_{s\in[0,1]}\mathcal{M}^0_{(J_s^\lambda, \phi_s^\lambda)}(x_0, y_0)$.

$$H\mathcal{M}(x_0, y_0) = \{(u, s) \in \mathcal{M}^0_{(J_s^\lambda, \phi_s^\lambda)}(x_0, y_0) \times [0,1]\} \subset \mathcal{P}(L, \phi_s^\lambda; P)(x_0, y_0) \times [0,1].$$

$H\mathcal{M}(x_0, y_0)$ are regular zero sets of $\overline{\partial}_{J(J_s^\lambda, \phi_s^\lambda)}$ and are smooth manifolds of dimension $\mu^{(r)}(y_0^{(r)}) - \mu^{(r)}(x_0^{(r)}) + 1$. Consider the case of $\mu^{(r)}(x_0^{(r)}) = \mu^{(r)}(y_0^{(r)}) = n$, the boundary of 1-dimensional sub-manifold $H\mathcal{M}(x_0, y_0)$ of $\mathcal{P}(L, \phi_s^\lambda; P)(x_0, y_0) \times [0,1]$ consists of

- $\mathcal{M}^0_{(J^\lambda, \phi^\lambda)}(x_0, y_0) \times \{0\} \cup \mathcal{M}^0_{(\overline{J}^\lambda, \overline{\phi}^\lambda)}(x_0, y_0) \times \{1\}$
- $\cup_{s\in[0,1],y}\mathcal{M}^{-1}_{(J_s^\lambda, \phi_s^\lambda)}(x_0, y) \times \mathcal{M}^0_{(J^2, \phi^2)}(y, y_0)$ for $y \in C_{n-1}^{(r)}(L, \phi^2; P, J^2)$.
- $\mathcal{M}^0_{(J^0, \phi^0)}(x_0, x) \times \cup_{s\in[0,1],x}\mathcal{M}^{-1}_{(J_s^\lambda, \phi_s^\lambda)}(x, y)$ for $x \in C_{n-1}^{(r)}(L, \phi^0; P, J^0)$.

Note that $\mathcal{M}^{-1}_{(J_s^\lambda, \phi_s^\lambda)}(x_0, y)$ and $\mathcal{M}^{-1}_{(J_s^\lambda, \phi_s^\lambda)}(x, y)$ are solutions of (u, s) of J-holomorphic equations lying in virtual dimension -1, they can only occur for $0 < s < 1$. Define $H : C_n^{(r)}(L, \phi^0; P, J^0) \to C_{n-1}^{(r)}(L, \phi^2; P, J^2)$ by

$$H(x_0) = \sum_y \sum_s \#\mathcal{M}^{-1}_{(J_s^\lambda, \phi_s^\lambda)}(x_0, y). \tag{3.11}$$

Similar to Lemma 3.3, by checking the corresponding preferred lifts and the integral Maslov index, we get the desired cochain homotopy H between ϕ_{02}^n and $\overline{\phi}_{02}^n$ satisfying (3.10). $\square$

From Lemma 3.5, we have that ϕ_{02}^* from $(J^\lambda, \phi^\lambda)$ is the same homomorphism of $\overline{\phi}_{02}^*$ from $(\overline{J}^\lambda, \overline{\phi}^\lambda)$ on the integer graded symplectic Floer cohomology. Then we can prove the invariance of the integer graded symplectic Floer cohomology under the continuation of (J, ϕ).

Theorem 3.6. *For any continuation $(J^\lambda, \phi^\lambda) \in \mathcal{P}_{1,\varepsilon/2}$ which is regular at the ends, there exists an isomorphism*

$$\phi_{02}^n : I_n^{(r)}(L, \phi^0; P, J^0) \to I_n^{(r)}(L, \phi^1; P, J^1),$$

for all $n \in \mathbf{Z}$.

Proof: Let $(J^{-\lambda}, \phi^{-\lambda})$ be the reversed family of $(J^{\lambda}, \phi^{\lambda})$ by $\tau = -\tau^{'}$. So we can form a family of composition $(J^{\lambda}, \phi^{\lambda})\#_T(J^{-\lambda}, \phi^{-\lambda})$ in $\mathcal{P}_{2,\varepsilon}$ for some fixed $T(>2)$. By Lemma 3.4, we have

$$\phi_{(J^{\lambda},\phi^{\lambda})\#_T(J^{-\lambda},\phi^{-\lambda})} = \phi_{10}^* \circ \phi_{01}^*.$$

For $(J^{\lambda}, \phi^{\lambda})\#_T(J^{-\lambda}, \phi^{-\lambda})$, it can be deformed to the trivial continuation as (J^0, ϕ^0) for all $\tau \in \mathbf{R}$. Then by Lemma 3.5, we have

$$\phi_{10}^* \circ \phi_{01}^* = \phi_{00}^* = id : I_*^{(r)}(L, \phi^0; P, J^0) \to I_*^{(r)}(L, \phi^0; P, J^0).$$

Similarly, $\phi_{01}^* \circ \phi_{10}^* = \phi_{11}^* = id$ on $I_*^{(r)}(L, \phi^1; P, J^1)$. Thus the result follows. $\square$

Now the integer graded symplectic Floer cohomology $I_*^{(r)}$ is functorial with respect to compositions of continuations $(J^{\lambda}, \phi^{\lambda})$ and invariant under continuous deformations of $(J^{\lambda}, \phi^{\lambda})$ within the set of continuations $\mathcal{P}_{1,\varepsilon/2}$.

4. Spectral sequence for the symplectic Floer cohomology

In this section we are going to show that the Z-graded symplectic Floer cohomology $I_*^{(r)}(L, \phi; P)$ for $r \in \mathbf{R}_{L,\phi}$ and $* \in \mathbf{Z}$ determines the $\mathbf{Z}_{\Sigma(L)}$-graded symplectic Floer cohomology $HF^*(L, \phi; P), * \in \mathbf{Z}_{\Sigma(L)}$. The way to link them together is to filter the integer graded Floer cochain complex. The filtration, by a standard method, formulates a spectral sequence which converges to the $\mathbf{Z}_{\Sigma(L)}$ graded symplectic Floer cohomology $HF^*(L, \phi; P)$.

Definition 4.1. For $r \in \mathbf{R}_{L,\phi}, j \in \mathbf{Z}_{\Sigma(L)}$ and $n \equiv j \pmod{\Sigma(L)}$, define the free Z_2 modules

$$F_n^{(r)} C_j(L, \phi; P, J) = \sum_{k \geq 0} C_{n+\Sigma(L)k}^{(r)}(L, \phi; P, J),$$

which gives a natural decreasing filtration on $C_*(L, \phi; P, J), * \in \mathbf{Z}_{\Sigma(L)}$.

There is a finite length decreasing filtration of $C_j(L, \phi; P, J)$, $j \in \mathbf{Z}_{\Sigma(L)}$:

$$\cdots \subset F_{n+\Sigma(L)}^{(r)} C_j(L, \phi; P, J) \subset F_n^{(r)} C_j(L, \phi; P, J) \subset F_{n-\Sigma(L)}^{(r)} C_j(L, \phi; P, J) \subset \cdots \subset C_j(L, \phi; P, J). \tag{4.1}$$

$$C_j(L, \phi; P, J) = \cup_{n \equiv j (mod \ \ \Sigma(L))} F_n^{(r)} C_j(L, \phi; P, J). \tag{4.2}$$

Note that the symplectic action is non-increasing along the gradient trajectories (J-holomorphic curves on $\Theta \times \mathbf{R}$), it follows that the coboundary map $\delta : F_n^{(r)} C_j(L, \phi; P, J) \to F_{n+1}^{(r)} C_{j+1}(L, \phi; P, J)$ (in Theorem 2.5) preserves the filtration in definition 4.1. Thus the $\mathbf{Z}_{\Sigma(L)}$ graded symplectic Floer cochain complex $(C_j(L, \phi; P, J), \delta)_{j \in \mathbf{Z}_{\Sigma(L)}}$ has a decreasing bounded filtration $(F_n^{(r)} C_*(L, \phi; P, J), \delta)$,

$$\begin{array}{ccccc} & \downarrow & \downarrow & & \downarrow \\ \cdots & \subset F_{n+\Sigma(L)}^{(r)} C_j(L, \phi; P, J) & \subset F_n^{(r)} C_j(L, \phi; P, J) & \subset \cdots & \subset C_j(L, \phi; P, J) \\ & \downarrow \partial^{(r)} & \downarrow \partial^{(r)} & & \downarrow \delta \\ \cdots & \subset F_{n+\Sigma(L)+1}^{(r)} C_{j+1}(L, \phi; P, J) & \subset F_{n+1}^{(r)} C_{j+1}(L, \phi; P, J) & \subset \cdots & \subset C_{j+1}(L, \phi; P, J) \\ & \downarrow & \downarrow & & \downarrow \end{array} . \tag{4.3}$$

Lemma 4.2. *(1) The cohomology of the vertical cochain subcomplex $F_n^{(r)}C_*(L,\phi;P,J)$ in the filtration (4.3) is $F_n^{(r)}I_j^{(r)}(L,\phi;P,J)$.*

(2) There is a natural bounded filtration for $\{I_*^{(r)}(L,\phi;P,J)\}_{*\in Z}$ the integer graded symplectic Floer cohomology,

$$\cdots F^{(r)}_{n+\Sigma(L)}HF^j(L,\phi;P,J)\subset F_n^{(r)}HF^j(L,\phi;P,J)\subset F^{(r)}_{n-\Sigma(L)}HF^j(L,\phi;P,J)\cdots\subset I_j^{(r)}(L,\phi;P,J),$$

where $F_n^{(r)}HF^j(L,\phi;P,J)=\ker(I_j^{(r)}(L,\phi;P,J))\to F^{(r)}_{n-\Sigma(L)}I_j^{(r)}(L,\phi;P,J))$.

Proof: The results follows from definition 4.1 and standard results in [32] Chapter 9. □

Theorem 4.3. *For $\Sigma(L)\geq 3$,*

(i) There is a spectral sequence $(E^k_{n,j},d^k)$ with

$$E^1_{n,j}(L,\phi;P,J)\cong I_n^{(r)}(L,\phi;P,J),\quad n\equiv j\pmod{\Sigma(L)},$$

and

$$E^\infty_{n,j}(L,\phi;P,J)\cong F_n^{(r)}HF^j(L,\phi;P,J)/F^{(r)}_{n+\Sigma(L)}HF^j(L,\phi;P,J).$$

(ii) The spectral sequence $(E^k_{n,j},d^k)$ converges to the $Z_{\Sigma(L)}$ graded symplectic Floer cohomology $HF_*(L,\phi;P,J)$, where

$$d^k:E^k_{n,j}(L,\phi;P,J)\to E^k_{n+\Sigma(L)k+1,j+1}(L,\phi;P,J).$$

Proof: (i) Note that

$$F_n^{(r)}C_j(L,\phi;P,J)/F^{(r)}_{n+\Sigma(L)}C_j(L,\phi;P,J)=C_n^{(r)}(L,\phi;P,J).$$

It is standard from [32] that there exists a spectral sequence $(E^k_{n,j},d^k)$ with E^1 term given by the cohomology of $F_n^{(r)}C_j(L,\phi;P,J)/F^{(r)}_{n+\Sigma(L)}C_j(L,\phi;P,J)$, so we have $E^1_{n,j}(L,\phi;P,J)\cong I_n^{(r)}(L,\phi;P,J)$ and $E^\infty_{n,j}(L,\phi;P,J)$ is isomorphic to the bigraded Z_2-module associated to the filtration $F^{(r)}$ of the Z-graded symplectic Floer cohomology $I_n^{(r)}(L,\phi;P,J)$.

(ii) Since the Lagrangian intersections are transversal and Z_ϕ is a finite set of intersections $L\cap\phi(L)$, so the filtration F is bounded and complete from (4.2). Thus the spectral sequence converges to the $Z_{\Sigma(L)}$ graded symplectic Floer cohomology. Note that the grading is unusual (jumping by $\Sigma(L)$ in n), we list the terms for $Z^k_{*,*},E^k_{*,*}$.

$$Z^k_{n,j}(L,\phi;P,J)=\{x\in F_n^{(r)}C_j(L,\phi;P,J)|\delta x\in F^{(r)}_{n+1+\Sigma(L)k}C_{j+1}(L,\phi;P,J)\}$$

$$E^k_{n,j}(L,\phi;P,J)=Z^k_{n,j}(L,\phi;P,J)/\{Z^{k+1}_{n+\Sigma(L),j}(L,\phi;P,J)+\delta Z^{k-1}_{n+(k-1)\Sigma(L)-1,j-1}(L,\phi;P,J)\}$$

$$Z^\infty_{n,j}(L,\phi;P,J)=\{x\in F_n^{(r)}C_j(L,\phi;P,J)|\delta x=0\}$$

$$E^\infty_{n,j}(L,\phi;P,J)=Z^\infty_{n,j}(L,\phi;P,J)/\{Z^\infty_{n+\Sigma(L),j}(L,\phi;P,J)+dZ^\infty_{n+(k-1)\Sigma(L)-1,j-1}(L,\phi;P,J)\}.$$

Thus δ induces the higher differential

$$d^k:E^k_{n,j}(L,\phi;P,J)\to E^k_{n+\Sigma(L)k+1,j+1}(L,\phi;P,J).$$

□

Theorem 4.4. *For $\Sigma(L) \geq 3$,*

(1) for any continuation $(J^\lambda, \phi^\lambda) \in \mathcal{P}_{1,\varepsilon/2}$ which is regular at ends, there exists an isomorphism

$$E^1_{n,j}(L, \phi^0; P, J^0) \cong E^1_{n,j}(L, \phi^1; P, J^1).$$

(2) For each $k \geq 1$, $E^k_{n,j}(L, \phi; P, J)$ are the symplectic invariant under continuous deformations of $(J^\lambda, \phi^\lambda)$ within the set of continuations.

Proof: Clearly (2) follows from (1) by Theorem 1 in [32] page 468. From Theorem 4.3, we have an isomorphism $E^1_{n,j}(L, \phi^0; P, J^0) \cong I^{(r)}_n(L, \phi^0; P, J^0)$, so there exists an isomorphism from §3, $I^{(r)}_n(L, \phi^0; P, J^0) \to I^{(r)}_n(L, \phi^1; P, J^1)$ which respects the filtration and induces an isomorphism on the E^1 term. $\square$

Now we can view that $E^k_{n,j}(L, \phi; P, J) = E^k_{n,j}(L, \phi; P)$, $E^1_{n,j}(L, \phi; P) = I^{(r)}_n(L, \phi; P)$, for $k \geq 1, r \in \mathbf{R}_{L,\phi}$, are new symplectic invariants provided $\Sigma(L) \geq 3$. All these new symplectic invariants should contain more information on $(P, \omega; L, \phi)$, they are also finer than $HF^*(L, \phi; P), * \in \mathbf{Z}_{\Sigma(L)}$ the usual Floer cohomology. In particular, the minimal k for which $E^k = E^\infty$ should be meaningful, denoted by $k(L)$. We will discuss the applications of these symplectic invariants on Lagrangian embedding, Maslov index and Hofer symplectic energy norm.

Corollary 4.5. *For $\Sigma(L) \geq 3, j \in \mathbf{Z}_{\Sigma(L)}$,*

$$\sum_{k \in Z} I^{(r)}_{j+\Sigma(L)k}(L, \phi; P) = HF^j(L, \phi; P)$$

if and only if all the differentials d^k in the spectral sequence $(E^k_{n,j}, d^k)$ are trivial (i.e. $k(L) = 1$).

In general, we see that $\sum_{k \in Z} I^{(r)}_{j+\Sigma(L)k}(L, \phi; P) \neq HF^j(L, \phi; P)$ for $j \in Z_{\Sigma(L)}$. $I^{(r)}_*(L, \phi; P)$ can be thought as an integer lift of the symplectic Floer cohomology $HF^*(L, \phi; P)$.

5. Applications and Remarks

5.1. **Hofer's energy and Chekanov's construction.** Hofer [14] introduced the notion of disjunction energy or displacement energy associated with a subset of symplectic manifold. Roughly, the Hofer's symplectic energy measures how large a variation of a (compactly supported) Hamiltonian function must be in order to push the subset off itself by a time-one map of corresponding Hamiltonian flow. Hofer showed that his symplectic energy of every open subset in standard symplectic vector space is nontrivial. For more geometric study of the Hofer's energy, we refer to [16] and [17].

Definition 5.1. Let $\mathcal{H}$ be the space of compactly supported functions on $[0, 1] \times P$. The Hofer's symplectic energy of a symplectic diffeomorphism $\phi : P \to P$ is defined by

$$E(\phi) = \inf\{\int_0^1 (\max_{x \in P} H(s, x) - \min_{x \in P} H(s, x))ds | \phi \text{ is a time one flow generated by } H \in \mathcal{H}\}.$$

$$e_H(L) = \inf\{E(\phi) : \phi \in Ham(P), L \cap \phi(L) = \emptyset \quad \text{empty set}\}.$$

Theorem 5.2 (Chekanov [5])**.** *If $E(\phi) < \sigma(L)$, L is rational and L intersects $\phi(L)$ transversally, then*

$$\#(L \cap \phi(L)) \geq SB(L; Z_2),$$

where $SB(L; Z_2)$ is the sum of Betti number with Z_2 coefficients. (I.e. $e_H(L) \geq \sigma(L)$.)

Remarks: (1) Polterovich [29] used Gromov's figure 8 trick and a refinement of Gromov's existence scheme of J-holomorphic disc to show that $e_H(L) \geq \frac{\sigma(L)}{2}$. Chekanov extends the result to $e_H(L) \geq \sigma(L)$ which is optimal for general Lagrangian sub-manifolds, see [5] §1 and [23] §6, and he asked whether Theorem 5.2 remains true if $E(\phi) = \sigma(L)$. This is the case corresponding to $r_0 \equiv 0$ $(mod \sigma(L))$ which is not an regular value for the symplectic action a. (2) Sikorav [31] showed that $e_H(T^m) \geq \sigma(T^m)$. Theorem 5.2 generalizes the Sikorav's result to all rational Lagrangian sub-manifolds.

In [5], Chekanov was able to use a restricted Floer cohomology in the study of Hofer's symplectic energy of rational Lagrangian sub-manifolds. Denote

$$\Omega_s = \{\gamma \in C^\infty([0,1], P) | \gamma(0) \in L, \gamma(1) \in \phi_s(L)\},$$

$$\Omega = \cup_{s\in[0,1]} \Omega_s \subset [0,1] \times C^\infty([0,1], P).$$

One may choose the anti-derivative of $Da_s(z)\xi$ as $a_s : \Omega_s \to \mathbf{R}/\sigma(L)Z$. Chekanov fixed a_0 with critical value 0. Pick $z_s \in L \cap \phi_s(L)$ such that modulo $\sigma(L)\mathbf{Z}$,

$$0 < a_s(z_s) = \min\{a_s(x)(mod \sigma(L)\mathbf{Z}) | x \in Z_{\phi_s}\} < \sigma(L).$$

This is possible since $a_s(x) \equiv 0(mod \sigma(L)\mathbf{Z})$ for all $x \in Z_{\phi_s}$ will contradict with $E(\phi) < \sigma(L)$. So we may take $r_0 \neq 0$ sufficiently small in $\mathbf{R}_{L,\phi_s} \cap (0, \sigma(L))$, say $0 < r_0 < \frac{1}{8}a_s(z_s)$. The condition $E(\phi) < \sigma(L)$ provides that there is a unique $x \in Z_{\phi_s}$ (the $x^{(r_0)}$) which correspond to the unique lifts in $(r_0, r_0 + \sigma(L))$,

$$0 < a_s(x) - a_s(z_s) < \sigma(L). \tag{5.1}$$

Under these restrictions, define C_s the free Z_2 module generated by $L \cap \phi_s(L)$, and a coboundary map $\partial_s \in End(C_s)$ (see below) such that $\partial_s \circ \partial_s = 0$, so $H^*(C_s, \partial_s)$ is well-defined for every $s \in [0,1]$. Note that with the unique lifting x, y in $(r_0, r_0 + \sigma(L))$ we have the Chekanov's restricted Floer homology in the integer grading.

Lemma 5.3. *For r_0 as above, we have $C_*^{(r_0)}(L, \phi_s; P, J) = C_s$. Let $\mathcal{M}_s(x,y)$ be the restricted moduli space of J-holomorphic curves as the set $\{u \in \mathcal{M}(L, \phi_s(L)) | u^*(\omega) = a_s(x) - a_s(y)\}$. So we have*

$$\partial^{(r_0)} x = \partial_s x = \sum \#\hat{\mathcal{M}}_s(x,y) y.$$

Proof: Note that $\mu_u = \mu^{(r_0)}(y) - \mu^{(r_0)}(x) = dim \mathcal{M}_s(x,y)$ for any $u \in \mathcal{M}_s(x,y)$. So y in the coboundary $\partial_s x$ is the element in $C_{n+1}^{(r_0)}(L, \phi_s; P, J)$; for any $u \in \mathcal{M}(x,y)$ in the coboundary of $\partial^{(r_0)} x$, we have $u^*(\omega) = a_s(x) - a_s(y)$ ([7] Proposition 2.3). For the unique lift in $(r_0, r_0 + \sigma(L))$, the choice of a_s makes that there is an one-to-one correspondence between $\mathcal{M}_s(x,y)$ and $\mathcal{M}(x,y)$ for $\mu^{(r_0)}(y^{(r_0)}) - \mu^{(r_0)}(x^{(r_0)}) = 1$. So the coboundary maps agree on the Z_2 coefficients. $\square$

Hence for the choice of the r_0 we have the identification between Chekanov's restricted cohomology $H^*(C_s, \partial_s)$ and our integer graded Floer cohomology $I^{(r_0)}(L, \phi_s; P)$.

For s sufficiently small, $x \in L \cap \phi_s(L)$ is also a critical point of the Hamiltonian function H_s of ϕ_s. Then the Maslov index is related to the usual Morse index of H_s in the following:

$$\mu^{(r_0)}(x^{(r_0)}) = \mu_{H_s}(x) - m. \tag{5.2}$$

See [1], [6], [15] and [30].

Proposition 5.4. *For r_0 as above, we assume that (i) L is monotone Lagrangian sub-manifold in P, (ii) $\Sigma(L) \geq 3$, (iii) $E(\phi) < \sigma(L)$ and (iv) L intersects $\phi(L)$ transversally. Then there is a natural isomorphism between*

$$I_*^{(r_0)}(L, \phi_s; P) \cong H^{*+m}(L; Z_2) \quad \text{for } * \in Z \text{ and } s \in [0,1].$$

Proof: For any $s, s^{'} \in [0,1]$ sufficiently close, by Theorem 3.6 we have

$$I_*^{(r_0)}(L, \phi_s; P) \cong I_*^{(r_0)}(L, \phi_{s'}; P). \tag{5.3}$$

For $s \in [0,1]$ sufficiently small, we have

$$I_*^{(r_0)}(L, \phi_s; P) \cong H^*(C_s, \partial_s) \cong H^{*+m}(L; Z_2). \tag{5.4}$$

The first isomorphism is given by Lemma 5.3 and the second by Lemma 3 in [5]. Then the result follows from finite steps of applying (5.3) and (5.4). $\square$

Remarks: (i) Proposition 5.4 provides the Arnold conjecture for monotone Lagrangian sub-manifold with $\Sigma(L) \geq 3$ and $E(\phi) < \sigma(L)$. Chekanov's result in [5] does not require the assumptions of the monotonicity of L and $\Sigma(L) \geq 3$. Floer proved the Arnold conjecture for monotone case. The generalized version has been obtained by Hofer and Salamon [15], and Ono [25] for weakly monotone case. As in [22] Remark 3.7, we can also use the integer graded Floer cohomology to show Gromov's non-exactness theorem for compact Lagrangian embeddings into C^n.

(ii) Using $I_*^{(r_0)}(L, \phi_s; P)$, there is a natural relation $I_*^{(r_0+\sigma(L))}(L, \phi_s; P) = I_{*+\Sigma(L)}^{(r_0)}(L, \phi_s; P)$ for different choices of $a_s, a_s + \sigma(L)$. In fact

$$I_*^{(r_0)}(L, \phi_s; P) \cong I_*^{(r_0)}(L, \phi_{s'}; P),$$

for $s, s^{'} \in [0,1]$ which answers Oh's question in [23] page 29.

For Lagrangian sub-manifold $L \subset (C^m, \omega_0)$ with the standard symplectic structure $\omega_0 = -d\lambda$, where λ is the Liouville form, there is a Liouville class $[\lambda|_L] \in H^1(L, \mathbf{R})$. One of the fundamental results in [12] is the non-triviality of the Liouville class.

Theorem 5.5 (Gromov [12])**.** *For any compact Lagrangian embedding L in C^m, the Liouville class $[\lambda|_L] \neq 0 \in H^1(L, \mathbf{R})$.*

In particular, $H^1(L, Z_2) \neq 0$.

Theorem 5.6. *For any small $\varepsilon > 0$, if $e_H(L) = \sigma(L) - \varepsilon$ for a monotone Lagrangian sub-manΦfold L embedded in C^m, then $\Sigma(L) \leq 2$.*

Proof: Suppose the contrary, $\Sigma(L) \geq 3$. By the hypothesis, there exists a $\phi \in \mathcal{D}_\omega$ such that

$$e_H(L) < E(\phi) \leq e_H(L) + \frac{\varepsilon}{2} < \sigma(L). \tag{5.5}$$

So the time one flow ϕ separates the L from its definition of $e_H(L)$, i.e. $L \cap \phi(L) = \emptyset$. By definition of integer graded symplectic Floer cohomology, we have $I_*^{(r_0)}(L, \phi; P) = 0$. By Proposition 5.4, we have

$$I_*^{(r_0)}(L, \phi_s; P) \cong H^{*+m}(L; Z_2) \qquad \text{for } s \in [0, 1].$$

This contradicts with Theorem 5.5 $H^*(L; Z_2) \neq 0$. So we obtain the restriction on the Maslov index. □

5.2. **Lagrangian rigidity and Audin's question.**

Proposition 5.7. *For any compact monotone Lagrangian sub-manifold in (P, ω), if $\Sigma(L) \geq m + 1, (m \geq 2)$, then*

1. *all the differentials d^k are trivial for $k \geq 0$,*
2. *we have the following relations.*

$$\sum_{k \in Z} I^{(r_0)}_{j+\Sigma(L)k}(L, \phi; P) = HF^j(L, \phi; P).$$

Proof: By Proposition 5.4, for s sufficiently small the Maslov index for $I_*^{(r_0)}(L, \phi_s; P, J)$ satisfies the following.

$$0 < \max \mu^{(r_0)}(y^{(r_0)}) - \min \mu^{(r_0)}(x^{(r_0)}) = \mu_{H_s}(y) - \mu_{H_s}(x) \leq m.$$

The result follows from the definition of d^k. □

Theorem 5.8 (Oh [22]). *For any compact monotone Lagrangian embedding $L \subset C^m$,*

$$1 \leq \Sigma(L) \leq m.$$

Note that the result is optimal based on examples in [27]. We can use the integer graded symplectic Floer cohomology to prove Theorem 5.8 by the same argument used for the Floer-Oh's local Floer cohomology. We define the associated Poincaré-Laurent polynomials (shifted in degree) for the spectral sequence $P(E^k, t), k \geq 1$ by

$$P^{(r)}(E^k, t) = \sum_{n \in Z} (\dim_{Z_2} E^k_{n,j}) t^n. \tag{5.6}$$

Note that our polynomials are slightly different from the one formulated in [22]. By Theorem 4.3, $P^{(r)}(E^1, t) = \sum_{n \in Z} (\dim_{Z_2} I_n^{(r)}(L, \phi; P, J)) t^n$. From §5.1 Remarks (ii), we have

$$P^{(r+\sigma(L))}(E^k, t) t^{\Sigma(L)} = P^{(r)}(E^k, t). \tag{5.7}$$

Proposition 5.9. *For monotone Lagrangian L with $\Sigma(L) \geq 3$,*

$$P^{(r)}(E^k, t) = \sum_{i=1}^{k} (1 + t^{-i\Sigma(L)-1})\overline{Q}_i(t) + P^{(r)}(HF^*, t),$$

where $k+1 = k(L)$ and $\overline{Q}_i(t)$ are Poincaré-Laurent polynomials of nonnegative integer coefficients.

Proof: Let $Z^1_{n,j} = \ker\{d^1 : E^1_{n,j} \to E^1_{n+\Sigma(L)+1,j+1}\}$ and $B^1_{n,j} = Imd^1 \cap E^1_{n,j}$. we have the exact sequences

$$0 \to Z^1_{n,j} \to E^1_{n,j} \to B^1_{n+\Sigma(L)+1,j+1} \to 0,$$
$$0 \to B^1_{n,j} \to Z^1_{n,j} \to E^2_{n,j} \to 0.$$

So the degree $\Sigma(L)+1$ of d^1 derives the followings.

$$P^{(r)}(E^1, t) = P^{(r)}(E^2, t) + (1 + t^{-\Sigma(L)-1})P^{(r)}(B^1, t). \tag{5.8}$$

Since the higher differential d^i has degree $i\Sigma(L)+1$, we can repeat (5.8) and let $\overline{Q}_i(t) = P^{(r)}(B^i, t)$. Note that $E^\infty \cong HF^*(L, \phi; P)$ by Theorem 3.6. Thus we obtain the desired result. □

For any oriented monotone Lagrangian torus in C^m, its Maslov number is always even. Suppose $\Sigma(L) \neq 2$. Then $\Sigma(L) \geq 4(> 3)$. From the construction of integer graded symplectic Floer cohomology, $I^{(r_0)}_*(L, \phi; P)$ is well-defined provided $\Sigma(L) \geq 3$. Also the Floer cohomology $HF^*(L, \phi; P)$ is well-defined and is invariant under the generic continuation of (J, ϕ) from [21]. By choosing a Hamiltonian isotope $\phi = \{\phi_t\}$ such that

$$L \cap \phi_1(L) = \emptyset,$$

which is certainly possible in C^m. Thus $Z_{\phi_1} = \emptyset$, and

$$HF^*(L, \phi; P) = 0. \tag{5.9}$$

On the other hand, while ϕ_s is sufficiently close to identity map with $E(\phi_s) < \sigma(L)$, by Proposition 5.4 we have

$$I^{(r_0)}_*(L, \phi_s; P) \cong H^{*+m}(L; Z_2).$$

Applying Proposition 5.9 for $E^1 = H^{*+m}(L; Z_2)$, $E^\infty = HF^*(L, \phi; P) = 0$ from (5.9), we get the relation

$$(1+t)^m t^{-m} = \sum_{i=1}^{k} (1 + t^{-i\Sigma(L)-1})\overline{Q}_i(t), \tag{5.10}$$

where $(1+t)^m t^{-m}$ is the Poincaré-Laurent polynomial for the torus T^m. Note that our Poincaré-Laurent polynomial (5.10) is similar to the one in [22], [23]. The changing $+1$ into -1 reflects precisely the degrees $\Sigma(L)+1, -\Sigma(L)+1$ of differentials in the spectral sequences for the Floer cohomology. After shifting the degree, we just have the following Poincaré polynomial:

$$(1+t)^m = \sum_{i=1}^{k} (1 + t^{i\Sigma(L)+1})Q_i(t), \tag{5.11}$$

where $Q_i(t) = t^{m-i\Sigma(L)-1}\overline{Q}_i(t)$ is a polynomial in t with nonnegative integer coefficients.

Lemma 5.10. *For an embedded, oriented, monotone Lagrangian torus L in C^m with $k(L) \leq 2$, we have*

$$\Sigma(L) = 2.$$

Proof: (i) If $k(L) = 1$, then $k = 0$, we get a contradiction of $HF^*(L, \phi; P) = 0$ and $E^\infty = I_*^{(r_0)}(L, \phi; P) \cong H^*(L; Z_2) \neq 0$ by Theorem 5.5.

(ii) If $k(L) = 2$, then by Proposition 5.9 and (5.11) we have the identity

$$(1+t)^m = (1 + t^{\Sigma(L)+1})Q_1(t).$$

But this is impossible to have such a decomposition of $(1+t)^m = (1+t^{\Sigma(L)+1})q_1(t)$ for any even $\Sigma(L) \geq 3$. So $\Sigma(L)$ has to be an even integer in $2 \leq \Sigma(L) < 3$, i.e. $\Sigma(L) = 2$. ☐

Remark: Note that the result can be restated as that for any given compatible almost complex structure J, the monotone Lagrangian torus L carry a J-holomorphic disc $u : (D^2, \partial D^2) \to (C^m, L)$ with $\mu_L(u) = 2$. Following the discussion in [22], one has $0 < k\Sigma(L) \leq m + 1$, i.e

$$1 \leq k(L) = k + 1 \leq [\frac{m+1}{\Sigma(L)}] + 1.$$

In particular, if $\Sigma(L) \geq 3$ and $\Sigma(L)|(m + 1)$, then the last term in (5.11) may be

$$(1 + t^{[\frac{m+1}{\Sigma(L)}]\Sigma(L)+1}Q_{[\frac{m+1}{\Sigma(L)}]}(t) = (1 + t^{m+2})Q_{[\frac{m+1}{\Sigma(L)}]}(t).$$

Comparing with the left hand side of (5.11), we have $Q_{[\frac{m+1}{\Sigma(L)}]}(t) = 0$, i.e. $k(L) = k+1 \leq (m+1)/\Sigma(L)$ in this case.

Theorem 5.11. *For an embedded, oriented, monotone Lagrangian torus L in C^m, we have*

$$\Sigma(L) = 2,$$

unless $k(L) = \frac{m+1}{\Sigma(L)}$.

Proof: Suppose the contrary. So $\Sigma(L) \geq 3$. Our integer graded symplectic Floer cohomology is well-defined for the transverse intersection $L \cap \phi(L)$. We have $I_*^{(r_0)}(L, \phi_s; P) \cong H^{*+m}(L; Z_2)$. So $E^1_{n,j}$ is contributed from the cohomology of torus with corresponding degree shift, i.e.

$$H^j(T^m; Z_2) = \oplus_{n \equiv j mod \Sigma(L)} E^1_{n,j}.$$

Note that our higher differentials

$$d^k : E^k_{n,j} \to E^k_{n+k\Sigma(L)+1, j+1},$$

from $H^j(T^m; Z_2)$ to $H^{j+1}(T^m; Z_2)$ modulo E^{k-1}. By (5.9), $HF^*(T^m, \phi; P, Z_2) = 0$, so there are no elements in E^* survived from the differentials. By counting the rank, we have

$$\sum_{j=0}^{\Sigma(L)-1} \sum_{i=1}^{k} (-1)^j rank(\oplus_{n \equiv j mod \Sigma(L)} E^i_{n,j}) = 0, \tag{5.12}$$

where $j = 0, 1, \cdots, \Sigma(L) - 1$. Clearly we have, for each $j = 0, 1, \cdots, \Sigma(L) - 1$,

$$\sum_{i=1}^{k} rank(\oplus_{n \equiv j mod \Sigma(L)} E^i_{n,j}) = rank H^j(T^m; Z_2)$$
$$+ rank H^{j+\Sigma(L)}(T^m; Z_2) + \cdots + rank H^{j+k\Sigma(L)}(T^m; Z_2).$$

Thus (5.12) gives us the following identity

$$\sum_{l=0}^{k} \left(\begin{array}{c} m \\ l\Sigma(L) \end{array} \right) - \sum_{l=0}^{k} \left(\begin{array}{c} m \\ l\Sigma(L)+1 \end{array} \right) + \cdots - \sum_{l=0}^{k} \left(\begin{array}{c} m \\ l\Sigma(L)+\Sigma(L)-1 \end{array} \right) = 0. \tag{5.13}$$

This can be rearrange into the alternating sum in the following due to the even number $\Sigma(L)$.

$$\sum_{l=0}^{(k+1)\Sigma(L)-1} (-1)^l \left(\begin{array}{c} m \\ l \end{array} \right) = 0. \tag{5.14}$$

This derives another constraint

$$(k+1)\Sigma(L) - 1 \leq m, \tag{5.15}$$

i.e. $(k+1) \leq \frac{m+1}{\Sigma(L)}$ which is better restriction than $(k+1) \leq \frac{m+1}{\Sigma(L)} + 1$. So we have $(k+1) \leq [\frac{m+1}{\Sigma(L)}]$.

(i) If $(k+1) \leq [\frac{m+1}{\Sigma(L)}] - 1$, then

$$\begin{aligned} (k+1)\Sigma(L) - 1 &\leq ([\frac{m+1}{\Sigma(L)}] - 1)\Sigma(L) - 1 \\ &\leq m - \Sigma(L), \end{aligned}$$

for $\Sigma(L) \geq 3$. Thus the equation (5.14) will be equivalent to

$$\sum_{l=(k+1)\Sigma(L)}^{m} (-1)^l \left(\begin{array}{c} m \\ l \end{array} \right) = 0, \tag{5.16}$$

by $(1-1)^m = 0$. This is impossible for $m \geq 3$ and (5.16) with $4 \leq (k+1)\Sigma(L) < m - 2$. So the contradiction shows that $\Sigma(L) = 2$ for this case.

(ii) If $k = [\frac{m+1}{\Sigma(L)}]$, $m + 1 = k\Sigma(L) + r$, $0 \leq r \leq \Sigma(L) - 1$, then by the discussion above, we have $k \leq \frac{m+1}{\Sigma(L)} - 1$ if $r = 0$. So we only consider $1 \leq r \leq \Sigma(L) - 1$. In this case, we obtain

$$(k+1)\Sigma(L) - 1 = m - r + \Sigma(L) \geq m + 1,$$

which contradicts with (5.15).

(iii) If $(k+1) = [\frac{m+1}{\Sigma(L)}]$ for $r \neq 0$ in

$$m + 1 = (k+1)\Sigma(L) + r.$$

It is easy to see that

$$(k+1)\Sigma(L) - 1 = m - r < m.$$

So by the same argument in (i), we have a contradiction. Therefore the result follows except the case $m + 1 = (k+1)\Sigma(L)$ (i.e. $k(L) = \frac{m+1}{\Sigma(L)}$). □

Since $\Sigma(L)$ is always even for an oriented Lagrangian sub-manifold L, we offered an affirmative answer to the Audin's question for all even dimension m by Theorem 5.11. Now we are going to

complete the proof of Theorem C by considering the odd dimension m ($m \equiv 1 \pmod 2$) with the divisibility $\Sigma(L)|(m+1)$.

For an embedded, oriented, monotone Lagrangian torus L in (C^m, ω), we have $I_\omega = \lambda I_{\mu,L}$ for $\lambda \geq 0$. It is not hard to see that $L \times L$ is monotone in $(C^m \times C^m, \omega \oplus \omega)$. For $u : (D^2, \partial D^2) \to (C^m \times C^m, L \times L)$, we denote $p_i : C^m \times C^m \to C^m$ to be the projection on the i-th factor, $i = 1, 2$. Thus one has

$$I_{\mu\oplus\mu, L\times L}(u) = I_{\mu,L}(p_1 u) + I_{\mu,L}(p_2 u),$$

which follows from the product symplectic form and the Künneth formula for the Maslov class in $\Lambda(C^m) \times \Lambda(C^m)$ (see [4]). Hence we have

$$\begin{aligned} I_{\omega\oplus\omega}(u) &= I_\omega(p_1 u) + I_\omega(p_2(u) \\ &= \lambda I_{\mu,L}(p_1 u) + \lambda I_{\mu,L}(p_2 u) \\ &= \lambda I_{\mu\oplus\mu, L\times L}(u). \end{aligned}$$

Proposition 5.12. *For m odd, $\Sigma(L)|(m+1)$, L is an oriented monotone Lagrangian embedding into C^m, we have $\Sigma(L) = 2$.*

Proof: Suppose the contrary $\Sigma(L) > 3$. Combining the discussion above, we have a monotone, oriented embedding

$$L \times L \hookrightarrow (C^m \times C^m, \omega \oplus \omega).$$

Note that $\Sigma(L \times L) \geq \Sigma(L)$ from the additivity of the Maslov index, we have $\Sigma(L \times L) > 3$, and $L \times L$ is an oriented monotone Lagrangian embedding into $2m$ **even** dimensional complex plane. Thus by Theorem 5.11, we have $2 = \Sigma(L \times L) \geq \Sigma(L) > 3$ which is absurd. Thus $\Sigma(L) = 2$ for the last case. ☐

Remark: Using the same method, we can also obtain the result from the Floer-Oh's local homology and the spectral sequence defined in [22]. The point is to use the full spectral sequence information, rather than just the identity from the Pincaré polynomial of the spectral sequence.

There are four interesting numbers of a monotone Lagrangian manifold $L, \Sigma(L), \sigma(L), e_H(L), k(L)$. Both of them intertwines together and links with the integer graded symplectic Floer cohomology and its derived spectral sequence. It is interesting to study further relations among them.

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